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Interchanging rows two and three by (1), changing signs in row two by (2), and reducing column two by (3):

$$\begin{Vmatrix} 1 & 0 & -12 & 0 & 0 & -1 \\ 0 & 1 & -7 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & 2 \end{Vmatrix}.$$

Changing the signs in the last row by (2) and reducing column three by (3):

$$\begin{Vmatrix} 1 & 0 & 0 & -36 & -12 & -25 \\ 0 & 1 & 0 & -22 & -7 & -15 \\ 0 & 0 & 1 & -3 & -1 & -2 \end{Vmatrix}.$$

Hence,

$$B^{-1} = \begin{Vmatrix} -36 & -12 & -25 \\ -22 & -7 & -15 \\ -3 & -1 & -2 \end{Vmatrix}.$$

CENTERS OF SIMILITUDE AND THEIR N-DIMENSIONAL ANALOGIES.

By BANCROFT HUNTINGTON BROWN, Brown University.

1. In the January, 1915, issue of the MONTHLY[1] "Centers of similitude of circles and certain theorems attributed to Monge" were discussed. The theorems there given are the following:

(A) *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*[2]

(B) *The vertices of the six common tangent cones of three spheres, taken in pairs, lie by threes on four straight lines.*

(C) *Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If the six other tangent cones be drawn, then their vertices lie by threes in planes*[3] *with threes of the first group.*

It was shown:

(1) That theorem (A) was, in all probability, known to the Greeks of two thousand years ago;

(2) That Fuss found, with regard to the *external* centers of similitude of coplanar circles, that:

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[1] R. C. Archibald, THE AMERICAN MATHEMATICAL MONTHLY, Vol. XXII, pp. 6–12.

[2] Symbolically, if $E_{m,n}$ denote the external, and $I_{m,n}$ the internal centers of similitude of the circles C_m and C_n ($m, n = 1, 2, 3$. $m \gtrless n$), the following groups of points are collinear:

$$E_{1,2},\ E_{1,3},\ E_{2,3};\ E_{1,2},\ I_{1,3},\ I_{2,3};\ I_{1,2},\ E_{1,3},\ I_{2,3};\ I_{1,2},\ I_{1,3},\ E_{2,3}.$$

[3] These planes have been called "planes of similitude."

(*D*) *For n coplanar circles, the* $n(n-1)/2!$ *external centers of similitude are situated, in general, by threes on* $n(n-1)(n-2)/3!$ *different straight lines;*[1]

(3) That the centers of similitude of the spheres of theorem (*C*) lie by sixes on eight different planes. Further generalization of these results does not seem to have been published. It is the object of this paper to give indications of such generalizations.

2. Consider first the number of centers of similitude of *n* coplanar circles; any circle combined with each of the $(n-1)$ remaining circles determines 2 centers of similitude, and since in this way each pair of centers of similitude is enumerated twice we arrive at the result:

n coplanar circles have, in general, $n(n-1)$ *centers of similitude.*[2]

Three circles may be selected from *n* circles in $n(n-1)(n-2)/3!$ different ways. With any such set of three circles we may by theorem (*A*) associate a group of four lines on which the centers of similitude lie by threes. Hence we have:

(*E*) *The* $n(n-1)$ *centers of similitude of n coplanar circles are situated, in general, by threes on* $4n(n-1)(n-2)/3!$ *different straight lines.*

Some particular cases may be noted: (*a*) If three (or more) circles have two common external tangents, then three (or more) of the centers of similitude coincide and *any* line through this point would contain at least three centers of similitude; moreover all of the centers of similitude lie on a single line. (*b*) Again, if two or more pairs of circles are in perspective[3] from the same point, then at least four axes of similitude coincide. For if $E_{1,2}$ and $E_{3,4}$ are coincident, this point is collinear with $E_{1,4}$ and $E_{2,4}$, with $E_{2,4}$ and $E_{2,3}$, and with $E_{1,3}$ and $E_{1,4}$; hence, these three lines coincide with the fourth line containing $E_{2,3}$, $E_{2,4}$, and $E_{3,4}$. Similar reasoning may be employed if $E_{1,2}$ and $I_{3,4}$ coincide. (*c*) If, of two pairs of equal circles, the lines of centers are parallel, the circles may be said to be in perspective from a point at infinity, in the direction of their lines of centers, and their external centers of similitude coincide at infinity. This is then a particular case of the preceding. (*d*) If three or more circles are tangent at one point, since this point of tangency is a center of similitude (external or internal) of any two of the circles, *any* line through this point contains at least three (coincident) centers of similitude; and, as in the case of every set of coaxial circles, all the centers of similitude are collinear. (*e*) If, of three circles, at least two are concentric, the six (or five) centers of similitude are collinear.

3. Monge was the first to give theorem (*C*) and he indicated five such planes:

$$E_{1,2},\ E_{1,3},\ E_{1,4},\ E_{2,3},\ E_{2,4},\ E_{3,4};\quad E_{1,2},\ E_{1,3},\ I_{1,4},\ E_{2,3},\ I_{2,4},\ I_{3,4};$$

$$E_{1,2},\ I_{1,3},\ E_{1,4},\ I_{2,3},\ E_{2,4},\ I_{3,4};\quad I_{1,2},\ E_{1,3},\ E_{1,4},\ I_{2,3},\ I_{2,4},\ E_{3,4};$$

$$I_{1,2},\ I_{1,3},\ I_{1,4},\ E_{2,3},\ E_{2,4},\ E_{3,4}.$$

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[1] These straight lines have been called "axes of similitude."

[2] If the circles (spheres) are equal and non-concentric, one center of similitude is at infinity, and the other bisects the line joining their centers. If two circles (spheres) are concentric we may say that the two centers of similitude coincide with the common center of the circles (spheres) or that one center of similitude coincides with the common center, the other being indeterminate.

[3] Any two circles are in perspective from either center of similitude.

There are three others, which he overlooked, making eight in all:

$I_{1,2}, I_{1,3}, E_{1,4}, E_{2,3}, I_{2,4}, I_{3,4}$; $I_{1,2}, E_{1,3}, I_{1,4}, I_{2,3}, E_{2,4}, I_{3,4}$; $E_{1,2}, I_{1,3}, I_{1,4}, I_{2,3}, I_{2,4}, E_{3,4}$.

That, in general, not more than six such centers are coplanar, and that not more than eight such planes exist, may be shown as follows: The six points in a plane are L's (I's or E's) with the six *different* pairs of suffixes. $E_{1,2}$ and $I_{1,2}$ cannot lie in one of these planes, for, if they did, the centers of spheres 1 and 2, and then of all the spheres would lie in this plane. Hence, we can determine all the planes by taking three L's with the same suffixes, *e. g.*, 1, 2, 1, 3, 1, 4; that is, in $2 \times 2 \times 2$, or eight, and only eight, different ways.

If now we consider n spheres, having $n(n-1)$ centers of similitude, we can choose 4 spheres in $n(n-1)(n-2)(n-3)/4!$ ways. And since with each of these sets of four spheres we can associate 8 planes containing 6 centers of similitude, we have

(*F*) *The* $n(n-1)$ *centers of similitude of* n *spheres lie, in general, by sixes, in* $8n(n-1)(n-2)(n-3)/4!$ *planes.*

4. Consider now five four-dimensional hyper-spheres. Four points $L_{i,j}$ determine a hyper-plane in which six other centers of similitude lie. For example, if we take $E_{1,2}$, $I_{1,3}$, $E_{1,4}$, $E_{1,5}$; applying theorem (*A*) to spheres and hyper-spheres, by considering the circles of plane sections, we know $E_{1,2}$, $I_{1,3}$, $I_{2,3}$ are collinear; also $E_{1,2}$, $E_{1,4}$, $E_{2,4}$; $E_{1,2}$, $E_{1,5}$, $E_{2,5}$; $I_{1,3}$, $E_{1,4}$, $I_{3,4}$; $I_{1,3}$, $E_{1,5}$, $I_{3,5}$; and $E_{1,4}$, $E_{1,5}$, $E_{4,5}$. Hence $E_{1,2}$, $I_{1,3}$, $E_{1,4}$, $E_{1,5}$, $I_{2,3}$, $E_{2,4}$, $E_{2,5}$, $I_{3,4}$, $I_{3,5}$, $E_{4,5}$ lie in one hyper-plane. By a proof similar to the one used in the case of spheres, (1) not more than 10 centers of similitude can, in general, lie in a hyper-plane, and (2) there can, in general, be 16, and only 16, such hyper-planes, since L_{ij} can be chosen in only $2 \times 2 \times 2 \times 2$ or 16 ways. We have then

(*G*) *The* 20 *external and internal centers of similitude of* 5 *four-dimensional hyper-spheres lie, in general, by tens in* 16 *hyper-planes.*

From among n hyper-spheres we may choose 5 in $n(n-1)(n-2)(n-3)(n-4)/5!$ different ways. And since each of these sets of hyper-spheres has the property of theorem (*G*), in general, there follows

(*H*) *The* $n(n-1)$ *centers of similitude of* n *four-dimensional hyper-spheres lie, in general, by tens in* $16n(n-1)(n-2)(n-3)(n-4)/5!$ *hyper-planes.*

5. We are now in a position to extend the investigation to $(p+1)$ p-dimensional hyper-spheres, lying in a p-dimensional space. Generalizing from the preceding theorems we see that $(p+1)p/2$ of the $(p+1)p$ centers of similitude should lie in a $(p-1)$-dimensional space. Further, there will be 2^p such spaces. Hence,

(*K*) *The* $(p+1)p$ *centers of similitude of* $(p+1)$ p*-dimensional hyper-spheres lie, in general, in* 2^p $(p-1)$*-dimensional spaces containing* $(p+1)p/2!$ *centers of similitude.*

Next, consider n p-dimensional hyper-spheres, where $n \geqq p+1$. We have

(*L*) *The* $n(n-1)$ *centers of similitude of* n p*-dimensional hyper-spheres lie,* *in general, in* $2^p\ n(n-1)\cdots(n-p)/(p+1)!$ $(p-1)$*-dimensional hyper-planes each containing* $(p+1)p/2!$ *centers of similitude.*

Finally, we may slightly generalize this result and arrive at

(M) *The* $n(n-1)$ *centers of similitude of* n m*-dimensional hyper-spheres lie, in general, in* $2^p\ n(n-1)\cdots(n-p)/(p+1)!$ $(p-1)$*-dimensional hyper-planes containing* $(p+1)p/2!$ *centers of similitude,* where $n > m \geqq p$.

6. For $(p+1)$ p-dimensional hyper-spheres there exists an interesting relation with regard to the number of exterior and interior centers in each $(p-1)$-dimensional hyper-plane.

We saw that for $p = 2$, there was one line containing 3 external, and 3 lines containing 1 external and 2 internal centers.

Monge showed for $p = 3$, that 1 plane contained 6 external centers, and 4 planes 3 external and 3 internal centers. The three which he omitted contain 2 external and 4 internal centers.

The results for $p = 4, 5, \cdots$ up to 10, may be readily obtained, and are sufficient to indicate a general relation. A display in tabular form will simplify the presentation. In the first table is shown the number of hyper-planes (planes, lines); and in the second is indicated, in a corresponding place, the number of external, and of internal centers of similitude contained in each corresponding hyper-plane (plane, line).

TABLE I.

$p = 2$	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
3	4	5	6	7	8	9	10	11
	3	10	15	21	28	36	45	55
			10	35	56	84	120	165
					35	126	210	330
							126	462

TABLE II.

$p = 2$	3	4	5	6	7	8	9	10
3, 0	6, 0	10, 0	15, 0	21, 0	28, 0	36, 0	45, 0	55, 0
1, 2	3, 3	6, 4	10, 5	15, 6	21, 7	28, 8	36, 9	45, 10
	2, 4	4, 6	7, 8	11, 10	16, 12	22, 14	29, 16	37, 18
			6, 9	9, 12	13, 15	18, 18	24, 21	31, 24
					12, 16	16, 20	21, 24	27, 28
							20, 25	25, 30

It will be observed that in the first table the numbers, with the exception of the last in the columns where p is odd, are the regular coefficients in the expansion of $(a+b)^{p+1}$; the last numbers, when p is odd, are half the middle coefficients of the expansion. The composition of the second table is not especially complicated, but perhaps it will be worth while to append a series of formulæ for the general case.

When p *is odd:*

Number of $(p-1)$-dimensional hyperplanes	External Centers of Similitude	Internal Centers of Similitude
1	$\frac{p^2+p}{2}$	0
$p+1$	$\frac{p^2-p}{2}$	p
$\frac{(p+1)p}{2!}$	$\frac{p^2-3p+4}{2}$	$(p-1)2$
$\frac{(p+1)p(p-1)}{3!}$	$\frac{p^2-5p+12}{2}$	$(p-2)3$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots(p-k+2)}{k!}$	$\frac{p^2-(2k-1)p+(k-1)2k}{2}$	$(p-(k-1))k$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+5}{2}\right)}{\left(\frac{p-1}{2}\right)!}$	$\frac{p^2-4p+3}{4}$	$\left(\frac{p+3}{2}\right)\left(\frac{p-1}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+3}{2}\right)}{2\cdot\left(\frac{p+1}{2}\right)!}$	$\frac{p^2-1}{4}$	$\left(\frac{p+1}{2}\right)^2$

When p *is even*, the last two lines become:

$\frac{(p+1)p(p-1)\cdots\left(\frac{p+6}{2}\right)}{\left(\frac{p-2}{2}\right)!}$	$\frac{p^2+8}{4}$	$\left(\frac{p+4}{2}\right)\left(\frac{p-2}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+4}{2}\right)}{\left(\frac{p}{2}\right)!}$	$\frac{p^2}{4}$	$\frac{p}{2}\left(\frac{p+2}{2}\right)$

HISTORICAL NOTE ON CENTERS OF SIMILITUDE OF CIRCLES.

By RAYMOND CLARE ARCHIBALD, Brown University.

In the paper on "Centers of Similitude of Circles and certain Theorems attributed to Monge. Were they known to the Greeks?"[1], I endeavored to show, especially through consideration of a problem in the book *On Tangencies*

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[1] American Mathematical Monthly, January, 1915, vol. 22, pp. 6–12. *Addenda:* In note 2, page 10, line 5, *for* cones *read* conics; on page 12, line 2, *for* side *read* sides. The two following historical notes may also be given:

(*a*) The theorem that "*The six centers of similitude of three coplanar circles lie by threes on four straight lines*" has been attributed to Monge who published it in 1798. Proof by analytical geometry that the three *external* centers of similitude are collinear was given by L. Puissant in his *Recueil de diverses propositions de géométrie*, Paris, 1801, pp. 50–56; it was probably here that the particular case of two of the circles being equal was first considered in recent times. No reference is made to Monge. There was a German edition by Hahn (Berlin, 1806). In the second French edition (Paris, 1809) mention is made (pp. 131–137) of Monge and the *general* theorem.